| CS255: Cryptography and Computer Security | Winter 2021 |
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# Assignment #1

Due: 11:59pm on Mon., Feb. 1, 2021, by Gradescope (each answer on a separate page)

**Problem 1.** The trouble with compression. Let $(E, D)$ be a semantically secure cipher that operates on messages in $\{0, 1\}^{\leq n}$ (i.e. messages whose length is at most $n$ bits). Suppose that the ciphertext output by the encryption algorithm is exactly 128 bits longer than the input plaintext. To reduce ciphertext size, there is a strong desire to combine encryption with lossless compression. We can think of compression as a function from $\{0, 1\}^{\leq n}$ to $\{0, 1\}^{\leq n}$ where, for some messages, the output is shorter than the input. As always, the compression algorithm is publicly known to everyone.

    **a.** Compress-then-encrypt: Suppose the encryptor compresses the plaintext message $m$ before passing it to the encryption algorithm $E$. Some $n$-bit messages compress well, while other messages do not compress at all. Show that the resulting system is not semantically secure by exhibiting a semantic security adversary that obtains advantage close to 1.

    **b.** Encrypt-then-compress: Suppose that instead, the encryptor applies compression to the output of algorithm $E$ (here you may assume the compression algorithm takes messages of length up to $n + 128$ bits as input). Explain why this proposal is of no use for reducing ciphertext size.

**Problem 2.** The movie industry wants to protect digital content distributed on DVDs. Here is one possible approach. Suppose there are at most a total of $n$ DVD players in the world (e.g. $n = 2^{32}$). We view these $n$ players as the leaves of a binary tree of height $\log_2 n$. Every node $v_j$ in this binary tree contains an AES key $k_j \in \mathcal{K}$. These keys are kept secret from consumers and are fixed for all time. At manufacturing time every DVD player is assigned a serial number $i \in \{0, \dots, n-1\}$. Let $S_i$ be the set of $1 + \log_2 n$ nodes along the path from the root of the binary tree to leaf number $i$. The manufacturer embeds in player number $i$ the $1 + \log_2 n$ keys associated with the nodes in $S_i$. In this way each DVD player ships with $1 + \log_2 n$ keys embedded in it, and these keys are supposedly inaccessible to the end user. A DVD movie $m$ is encrypted as

$$DVD := \underbrace{E(k_{\text{root}}, k)}_{\text{header}} \;\Big\|\; \underbrace{E(k, m)}_{\text{body}},$$

where $k \xleftarrow{\text{R}} \mathcal{K}$ is a fresh random key called a content key. Since all DVD players have the key $k_{\text{root}}$, all players can decrypt the content $m$. We refer to $E(k_{\text{root}}, k)$ as the header and $E(k, m)$ as the body. In what follows the DVD header may contain multiple ciphertexts where each ciphertext is the encryption of the content key $k$ under some key $k_i$ in the binary tree.

    **a.** Suppose the $1 + \log_2 n$ keys embedded in DVD player number $r$ are exposed by hackers and published on the Internet. Show that when the movie industry is about to distribute a new movie $m$, they can encrypt $m$ using a header containing $\log_2 n$ short ciphertexts, so that all

DVD players can decrypt the movie except for player number $r$. In effect, the movie industry disables player number $r$.

Hint: the header will contain $\log_2 n$ ciphertexts where each ciphertext is the encryption of the content key $k$ under certain $\log_2 n$ keys from the binary tree.

**b.** Next, suppose the keys embedded in $s$ DVD players $R = \{r_1, \ldots, r_s\}$ are exposed by hackers, where $s > 1$. Show that the movie industry can encrypt the contents of a new DVD using a header containing $O(s \log n)$ short ciphertexts so that all players can decrypt the movie except for the players in $R$. You have just shown that all hacked players can be disabled without affecting other consumers.

Side note: the AACS system used to encrypt Blu-ray and HD-DVD disks uses a related system. It was quickly discovered that hackers can expose player secret keys faster than the MPAA can revoke them.

**Problem 3.** The purpose of this problem is to clarify the concept of *advantage*. Consider the following two experiments EXP(0) and EXP(1):

- In EXP(0) the challenger flips a fair coin (probability $1/2$ for HEADS and $1/2$ for TAILS) and sends the result to the adversary $\mathcal{A}$.
- In EXP(1) the challenger always sends TAILS to the adversary.

The adversary's goal is to distinguish these two experiments: at the end of each experiment the adversary outputs a bit 0 or 1 for its guess for which experiment it is in. For $b = 0, 1$ let $W_b$ be the event that in experiment $b$ the adversary output 1. The adversary tries to maximize its distinguishing advantage, namely the quantity

$$\text{Adv} := \left| \Pr[W_0] - \Pr[W_1] \right| \quad \in [0, 1] \ .$$

The advantage Adv captures the adversary's ability to distinguish the two experiments. If the advantage is 0 then the adversary behaves exactly the same in both experiments and therefore does not distinguish between them. If the advantage is 1 then the adversary can tell perfectly what experiment it is in. If the advantage is negligible for all efficient adversaries (as defined in class) then we say that the two experiments are indistinguishable.

a. Calculate the advantage of each of the following adversaries:
- $\mathcal{A}_1$: Always output 1.
- $\mathcal{A}_2$: Ignore the result reported by the challenger, and randomly output 0 or 1 with equal probability.
- $\mathcal{A}_3$: Output 1 if HEADS was received from the challenger, else output 0.
- $\mathcal{A}_4$: Output 0 if HEADS was received from the challenger, else output 1.
- $\mathcal{A}_5$: If HEADS was received, output 1. If TAILS was received, randomly output 0 or 1 with equal probability.

b. What is the maximum advantage possible in distinguishing these two experiments? Explain.

**Problem 4.** Let $F$ be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$, where $\mathcal{K} = \mathcal{X} = \mathcal{Y} = \{0,1\}^n$.

    **a.** Show that $F_1\big((k_1, k_2), (x_1, x_2)\big) := F(k_1, x_1) \oplus F(k_2, x_2)$ is not a secure PRF. That is, construct a PRF adversary $\mathcal{A}_1$ that has non-negligible advantage in distinguishing $F_1(k, \cdot)$ from a random function in $\text{Funs}[\mathcal{X}^2, \mathcal{Y}]$. Hint: your attacker will query $F_1$ at four points.

    **b.** Show that $F_2(k, x) := F(k, x) \parallel F(k, F(k, x))$ is not a secure PRF. Here $\parallel$ means the concatenation of the two outputs. Hint: your adversary $\mathcal{A}_2$ will query $F_2$ at two inputs, where the second query depends on the answer to the first query.

    **c.** Prove that $F_3(k, x) := F(k, x) \oplus x$ is a secure PRF. Do so by proving the contrapositive: show that if an adversary $\mathcal{A}_3$ can distinguish $F_3(k, \cdot)$ from a random function then there is adversary $\mathcal{B}$ (that is a wrapper around $\mathcal{A}_3$) that can distinguish $F$ from a random function. This $\mathcal{B}$ will play the role of challenger to $\mathcal{A}_3$, and attack $F$.

**Problem 5.** Let $\pi : \mathcal{X} \to \mathcal{X}$ be a fixed public one-to-one function, where $\mathcal{X} := \{0,1\}^n$, and where $\pi$ and $\pi^{-1}$ are efficiently computable. The Even-Mansour pseudorandom permutation $(E, D)$ derived from $\pi$, is defined as

$$E\big((k_0, k_1), x\big) := \pi(x \oplus k_0) \oplus k_1 \quad \text{and} \quad D\big((k_0, k_1), c\big) := \pi^{-1}(c \oplus k_1) \oplus k_0.$$

This permutation corresponds to one round of AES, and can be shown to be a secure PRP when $\pi$ is chosen at random from $\text{Perms}[\mathcal{X}]$, and when $|\mathcal{X}|$ is sufficiently large. Let's look at some broken variants of Even-Mansour.

    **a.** Show that $E_1(k_1, x) := \pi(x) \oplus k_1$ is not a secure PRP.

    **b.** Show that $E_2(k_0, x) := \pi(x \oplus k_0)$ is not a secure PRP.

**Problem 6.** Exercising the definition of semantic security. Let $(E, D)$ be a semantically secure cipher defined over $(\mathcal{K}, \mathcal{M}, \mathcal{C})$, where $\mathcal{M} = \mathcal{C} = \{0,1\}^L$. Which of the following encryption algorithms yields a semantically secure scheme? Either give an attack or provide a security proof. To prove security, prove the contrapositive, that is prove that a semantic security attacker $\mathcal{A}$ on the proposed system gives a semantic security attacker $\mathcal{B}$ on $(E, D)$, with the same advantage.

    **a.** $E_1(k, m) := 0 \parallel E(k, m)$

    **b.** $E_2(k, m) := E(k, m) \parallel \text{parity}(m)$

    **c.** $E_3(k, m) := \text{reverse}(E(k, m))$

    **d.** $E_4(k, m) := E(k, \text{reverse}(m))$

Here, for a bit string $s$, $\text{parity}(s)$ is 1 if the number of 1's in $s$ is odd, and 0 otherwise; also, $\text{reverse}(s)$ is the string obtained by reversing the order of the bits in $s$, e.g., $\text{reverse}(1011) = 1101$.

**Problem 7.** Let $\mathcal{E} = (E, D)$ be a cipher. Consider the cipher $\mathcal{E}_2 = (E_2, D_2)$, where $E_2(k, m) = E(k, E(k, m))$. One would expect that if encrypting a message once with $E$ is secure then encrypting it twice as in $E_2$ should be no less secure. However, that is not always true.

    **a.** Show that there is a semantically secure cipher $\mathcal{E}$ such that $\mathcal{E}_2$ is not semantically secure.

**b.** Prove that for every CPA secure cipher $\mathcal{E}$, the cipher $\mathcal{E}_2$ is also CPA secure. As usual prove the contrapositive: for every CPA adversary $\mathcal{A}$ attacking $\mathcal{E}_2$ there is a CPA adversary $\mathcal{B}$ attacking $\mathcal{E}$ with about the same advantage and running time as $\mathcal{A}$. Adversary $\mathcal{B}$ uses $\mathcal{A}$ as a black box – it plays the role of CPA challenger to $\mathcal{A}$ with respect to $\mathcal{E}_2$. It uses $\mathcal{A}$ to win the CPA game against its own challenger with respect to $\mathcal{E}$.

**Problem 8.** Feedback. We would love to hear your feedback on how the course is going.

**a.** How long did you spend on this assignment?

**b.** Do you have any feedback on the course material, office hours, or any other topics? How can the teaching team support you better?