

Uniqueness of the $S(5,8,24)$ Steiner system

Theodore Hwa

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1 Introduction

A Steiner $S(5, 8, 24)$ system is a set X with $|X| = 24$ together with a set of subsets of size 8 of X , called *octads*, such that every subset of X of size 5 is contained in exactly one octad.

We show that the $S(5, 8, 24)$ Steiner system is uniquely determined up to a permutation of S_{24} . Throughout the proof, we'll see how the ideas in Curtis' Miracle Octad Generator¹ arise naturally.

Throughout, we assume that a $S(5, 8, 24)$ exists, and deduce enough properties about it to show its uniqueness. It is immediate that the number of octads must be $\binom{24}{5} / \binom{8}{5} = 759$.

2 Leech triangle

For any $x \in X$, if we take the set of octads containing x , and then remove x from those octads, we get a $S(4, 7, 23)$ Steiner system. Therefore, the number of octads containing any element x is $\binom{23}{4} / \binom{7}{4} = 253$.

Similarly, if we take any k elements of X , where $k \leq 5$, the number of octads containing those k elements is $\binom{24-k}{5-k} / \binom{8-k}{5-k}$.

If k is 6, 7, or 8, then number of octads containing k elements is either 0 or 1, by the definition of a Steiner system. However, if we assume that the elements were drawn from a fixed octad to begin with, then the answer is 1. We have proved the following result.

¹For those who are already familiar with the MOG, we note that we will use Conway's convention for the MOG (as described in Sphere Packings, Lattices, and Groups (SPLAG), Ch. 11), rather than Curtis' original convention (which differs by interchanging the last two columns).

Proposition 2.1 *Fix an octad \mathcal{O}_0 and an integer $0 \leq k \leq 8$. The number of octads containing any k particular elements of \mathcal{O}_0 is $\binom{24-k}{5-k}/\binom{8-k}{5-k}$ if $k \leq 5$, and 1 if $k > 5$.*

An important note is that the answer is independent of both the octad \mathcal{O}_0 and also which k elements of \mathcal{O}_0 we select. It only depends on k .

We may generalize this question as follows. Given any two disjoint subsets X_1 and X_2 of \mathcal{O}_0 , how many octads contain all elements of X_1 and no elements of X_2 ? Denote this number by λ_{X_1, X_2} .

The proposition above is the case where X_2 is empty. In the general case, the number can be computed inductively on the size of X_2 .

Proposition 2.2 *Let X_1, X_2 be two disjoint subsets of an octad. Fix $x \in X_2$. Then we have the relation*

$$\lambda_{X_1, X_2} = \lambda_{X_1, X_2 \setminus \{x\}} - \lambda_{X_1 \cup \{x\}, X_2 \setminus \{x\}}.$$

Proof. Rewrite the equation as

$$\lambda_{X_1, X_2 \setminus \{x\}} = \lambda_{X_1, X_2} + \lambda_{X_1 \cup \{x\}, X_2 \setminus \{x\}}.$$

Among the octads counted by the left side, the first term on the right side counts those not containing x , and the second term on the right side counts those containing x . \square

From the proposition, we see that the value of λ_{X_1, X_2} depends only on the cardinality of X_1 and X_2 , and therefore we may simply write $\lambda_{k,l}$ where k and l are nonnegative integers with $k+l \leq 8$. Putting together the above results, we have the following theorem, giving an array of numbers called the Leech triangle.

Theorem 2.3 *(The Leech Triangle) Recursively define an array of numbers $\lambda_{k,l}$ as follows, where k and l are nonnegative integers with $k+l \leq 8$. We set $\lambda_{k,0} = \binom{24-k}{5-k}/\binom{8-k}{5-k}$ if $k \leq 5$, and 1 if $k > 5$. If $l > 0$, we define*

$$\lambda_{k,l} = \lambda_{k,l-1} - \lambda_{k+1,l-1}.$$

Then $\lambda_{k,l}$ is the number octads containing any k elements and excluding any l disjoint elements of a fixed octad.

The traditional arrangement of the Leech triangle is as follows, where the number $\lambda_{j,i-j}$ is placed as the i th row and j th entry in the row (counting from 0 in both cases).

759									
506	253								
330	176	77							
210	120	56	21						
130	80	40	16	5					
78	52	28	12	4	1				
46	32	20	8	4	0	1			
30	16	16	4	4	0	0	1		
30	0	16	0	4	0	0	0	1	

Corollary 2.4 *The intersection of any two octads has size 0, 2, 4 or 8. In particular, the intersection always has even size.*

Proof. This follows from the last line of the Leech Triangle. \square

3 Sextets

Definition: A *tetrad* is any subset of X of size 4.

Let $1, 2, 3, 4$ be a fixed tetrad. If a is any other element of X , then $1, 2, 3, 4, a$ can be extended to a unique octad $1, 2, 3, 4, a, b, c, d$. If A is any element of X other than these 8 elements, then $1, 2, 3, 4, A$ can be extended to a unique octad $1, 2, 3, 4, A, B, C, D$. The elements a, b, c, d cannot be equal to any of A, B, C, D because otherwise the two distinct octads would have 5 elements in common. Therefore, X is partitioned into 6 tetrads, one of which is the original $1, 2, 3, 4$. Such a configuration is called a *sextet*.

It is convenient to fix a sextet, and arrange the 24 elements of X as a rectangle with 6 columns of 4, where each column is a tetrad of the sextet. We will use this language from now on, referring to the elements of our fixed sextet as *columns*. We assume that the original tetrad $1, 2, 3, 4$ is the first column. We denote this column by \mathcal{C}_1 .

So far, we only know that the union of the first column with any other column forms an octad. In fact, the union of *any* two columns forms an octad. That result is the goal of this section, but first we need some preliminary results.

Proposition 3.1 *If \mathcal{O} is any octad, then the intersection of \mathcal{O} with each of the 6 columns have the same parity.*

Proof. We showed in the previous section, that the intersection of any two octads has even size. Therefore, if \mathcal{C} is any column other than \mathcal{C}_1 , then

$|\mathcal{O} \cap (\mathcal{C} \cup \mathcal{C}_1)|$ is even. Thus $|\mathcal{O} \cap \mathcal{C}|$ and $|\mathcal{O} \cap \mathcal{C}_1|$ have the same parity. Since \mathcal{C} is arbitrary but \mathcal{C}_1 is fixed, the result follows. \square

We classify octads as *odd* or *even* according to the parity of their intersection with the 6 columns. Note that this is not an absolute classification of octads, but relative to our fixed sextet.

Corollary 3.2 *An odd octad has distribution 3,1,1,1,1,1 among the columns in some order.*

Proof. That is the only way that 6 odd non-negative integers can add up to 8. \square

Corollary 3.3 *Given elements 1,2,3 of X in the first column, an element a in another column, and an element A in a third column, then the unique octad containing 1,2,3, a , A is an odd octad and hence distributed 3,1,1,1,1,1 among the columns.*

Proof. Let \mathcal{O} be the octad containing 1,2,3, a , A . Then \mathcal{O} cannot contain the fourth element of the first column, for then the intersection of \mathcal{O} with the octad consisting of the first two columns would be at least 5, hence it would be 8, which is impossible since \mathcal{O} contains an element in the third column. Hence $|\mathcal{O} \cap \mathcal{C}_1| = 3$, so \mathcal{O} is an odd octad. \square

Theorem 3.4 *The union of any 2 columns is an octad.*

Proof. Let a, b, c, d and A, B, C, D be two columns \mathcal{C}_2 and \mathcal{C}_3 other than 1,2,3,4. Let \mathcal{O} be the unique octad containing a, b, c, d, A . We must show that \mathcal{O} is the octad a, b, c, d, A, B, C, D .

We know that \mathcal{O} must be an even octad, and it cannot contain any of 1,2,3,4, since otherwise its intersection with the octad 1,2,3,4, a, b, c, d would be too large. So if \mathcal{O} is not the desired octad a, b, c, d, A, B, C, D , then it must be $a, b, c, d, A, B, \alpha, \beta$ where α and β are from a fourth column \mathcal{C}_4 .

Let x be one of a, b, c , and let \mathcal{O}_x be the octad determined by 1,2,3, x , A . By the previous corollary, $\mathcal{O}_x \cap \mathcal{C}_4$ must have exactly one element. We cannot have $\mathcal{O}_x \cap \mathcal{C}_4 = \mathcal{O}_y \cap \mathcal{C}_4$ for distinct x and y , since otherwise the distinct octads \mathcal{O}_x and \mathcal{O}_y would intersect in at least 5 elements. Therefore, for some x , $\mathcal{O}_x \cap \mathcal{C}_4$ must be either α or β . But then $\mathcal{O}_x \cap \mathcal{O}$ would have size 3 (x , A , and either α or β), which is impossible. \square

Corollary 3.5 *(i) Every tetrad is contained in a unique sextet, and the union of any two tetrads of a sextet is an octad.*

(ii) If two octads have intersection size 4, then their symmetric difference is an octad.

(iii) An even octad has one of the two column distributions 4,4,0,0,0,0 or 2,2,2,2,0,0.

Proof. (i) is clear because any sextet can be taken to be the 6 columns. (ii) follows from (i) by considering the tetrad formed by the intersection of the two octads. (iii) follows because the previous proof rules out the distribution 4,2,2,0,0,0. \square

4 Hexacode

We fix a sextet and arrange its tetrads in 6 columns of 4, as in the previous section, thus getting a classification into odd and even octads according to the parity of the intersection of an octad with the 6 columns. We label the rows with the elements of \mathbb{F}_4 , the finite field of order 4: $0, 1, \omega, \bar{\omega}$.² In this section, we study the odd octads in detail.

Let \mathcal{O} be an *odd* octad. We know that the distribution of \mathcal{O} is 3, 1, 1, 1, 1, 1. To each column, we associate an element of \mathbb{F}_4 as follows. If \mathcal{O} contains 1 element from the column, we associate the row label for that element; if \mathcal{O} contains 3 elements of the column, we associate the row label for the missing element. Thus there is a mapping from the set of odd octads to \mathbb{F}_4^6 . We define the *hexacode*, \mathcal{H} , to be the image of this mapping. Note that \mathcal{H} depends on the choice of sextet and also on the ordering of the elements within each tetrad (i.e. column).

Proposition 4.1 (i) $|\mathcal{H}| = 64$.

(ii) An element of \mathcal{H} is uniquely determined by any 3 of the 6 positions.

(iii) An odd octad is uniquely determined by specifying an element of \mathcal{H} and which column contains 3 elements. Thus the number of odd octads is $64 \cdot 6 = 384$.

(iv) Any two distinct elements of \mathcal{H} agree in exactly 0 or 2 positions.

Proof. First consider the odd octads which have 3 elements in the first column. They have distribution 3,1,1 in the first 3 columns in that order. Those 5 elements uniquely determine an octad. There are 64 ways to select those 5 elements, and the images of the corresponding octads in \mathcal{H} are

²Of course, this choice of symbols is made with the benefit of hindsight. For most of this section, however, the field structure is irrelevant and we can regard these as arbitrary labels.

distinct (because the image of any two are distinct in the first 3 positions). Hence we have at least 64 elements in \mathcal{H} .

Next we consider an odd octad \mathcal{O} with 3 elements in some other column \mathcal{C} . By taking the symmetric difference of \mathcal{O} with the octad $\mathcal{C}_1 \cup \mathcal{C}$, we get an octad \mathcal{O}' with 3 elements in the first column. It is easily seen that \mathcal{O} and \mathcal{O}' have the same image in \mathcal{H} . Therefore \mathcal{H} has exactly 64 elements. The statements (ii) and (iii) are also clear from the above argument.

To show (iv), we know from (ii) that two distinct elements agree in 0,1, or 2 positions. If they agreed in exactly one position, then from (iii), consider the octads corresponding to them that have 3 elements in that position. Those two octads would have intersection size 3, which is a contradiction.

Theorem 4.2 *\mathcal{H} is uniquely determined up to permutation of the columns and a permutation of \mathbb{F}_4 within each column.*

Proof.

From (ii) of the previous proposition, we know there is an element of \mathcal{H} starting with $(0, 0, 0)$, so by applying a suitable permutation of \mathbb{F}_4 to each of the last three columns, we may assume that:

$$(0, 0, 0, 0, 0, 0)$$

is in \mathcal{H} . Next, considering the elements of \mathcal{H} starting with $(0, 0, x)$, since distinct elements of \mathcal{H} cannot agree in more than 2 positions (part (iv) of the previous proposition), we can further permute each of the last three columns so that \mathcal{H} contains:

$$(0, 0, 1, 1, 1, 1)$$

$$(0, 0, \omega, \omega, \omega, \omega)$$

$$(0, 0, \bar{\omega}, \bar{\omega}, \bar{\omega}, \bar{\omega})$$

Next consider $(0, 1, 0, ?, ?, ?)$. By comparison with the previous elements, we know that none of the last 3 positions can be 0 and also that they must be distinct, so they must be $1, \omega, \bar{\omega}$ in some order. Since the elements of \mathcal{H} so far are invariant under permutation of the last 3 columns, we may assume that \mathcal{H} contains:

$$(0, 1, 0, 1, \omega, \bar{\omega})$$

Next consider $(0, \omega, 0, ?, ?, ?)$ and $(0, \bar{\omega}, 0, ?, ?, ?)$. The unknown positions can only be $(\omega, \bar{\omega}, 1)$ and $(\bar{\omega}, 1, \omega)$. By swapping ω and $\bar{\omega}$ in the second position if necessary, we may assume that \mathcal{H} contains:

$$(0, \omega, 0, \omega, \bar{\omega}, 1)$$

$$(0, \bar{\omega}, 0, \bar{\omega}, 1, \omega)$$

Next consider the elements $(x, 0, 0, ?, ?, ?)$ where $x \neq 0$. The unknown elements must be $(1, \bar{\omega}, \omega)$ and its cyclic permutations. By permuting $1, \omega, \bar{\omega}$ in the first column if necessary, we may assume that \mathcal{H} contains:

$$(1, 0, 0, 1, \bar{\omega}, \omega)$$

$$(\omega, 0, 0, \omega, 1, \bar{\omega})$$

$$(\bar{\omega}, 0, 0, \bar{\omega}, \omega, 1)$$

Now we may verify that all remaining triples (x, y, z) have only one possible completion forced by these 10 elements. It is the standard hexacode, which miraculously turns out to be linear over \mathbb{F}_4 . This verification may be easily carried out by computer, but there are enough symmetries that it can also be carried out by hand. We note that the 10 elements we put in \mathcal{H} so far are closed under these symmetries: (1) multiplication by ω ; (2) swapping ω and $\bar{\omega}$ followed by swapping the first 2 columns; (3) swapping ω and $\bar{\omega}$ followed by swapping the last 2 columns. Hence it suffices to verify just one representative triple (x, y, z) per orbit.

5 Conclusion

A fixed sextet determines the even octads of distribution 4,4,0,0,0,0 (the union of any two columns), and together with the hexacode (which we proved is uniquely determined up to permutation), the odd octads are also uniquely determined (by part (iii) of the last theorem). We must show that the remaining octads, which are the even octads of distribution 2,2,2,2,0,0, are also uniquely determined.

This is straightforward. Suppose we have a 2,2,2,2,0,0 octad \mathcal{O} consisting of $1, 2, a, b, A, B, \alpha, \beta$. Let \mathcal{O}_1 be the octad containing $1, 2, 3, a, A$. Then \mathcal{O}_1 must be an odd octad, and it already has 4 known elements intersecting \mathcal{O} , so there are no others (since clearly $\mathcal{O} \neq \mathcal{O}_1$). Hence $|\mathcal{O} \cap \mathcal{O}_1| = 4$ so that $\mathcal{O}_2 = \mathcal{O} + \mathcal{O}_1$ (here $+$ denotes symmetric difference) is also an octad. The octad \mathcal{O}_2 consists of $3, b, B, \alpha, \beta, \gamma$ and two more elements, one in each of the last two columns. So \mathcal{O}_2 is also an odd octad. Thus every 2,2,2,2,0,0 octad $\mathcal{O} = \mathcal{O}_1 + \mathcal{O}_2$ is the sum of two odd octads. Since the odd octads are determined, then the 2,2,2,2,0,0 octads are also determined.

This proves that the Steiner system $S(5, 8, 24)$ is unique up to permutation of the 24 elements.