The Simplest Derivation of $E = mc^2$

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Introduction

The principle of mass-energy equivalence is one of the most profound insights in physics, revealing that matter is simply another form of energy. Yet, more than a century after its discovery, this concept remains unfamiliar to the general public. This disconnect may stem from the lack of clear and comprehensive explanations that make the principle accessible beyond academic settings. Simplified derivations of the mass-energy equation can help bridge this gap, making the underlying concepts more approachable and understandable for a broader audience.

Derivation of $E = mc^2$

Maxwell's electromagnetic equations have demonstrated that light travels at a constant speed, denoted as *c* (approximately 3×10^8 m/s), which has been confirmed through numerous experiments and observations. Additionally, the momentum (*p*) of electromagnetic waves, or photons, is proportional to their energy level (*E*):

(1) $$p = \frac{E}{c}$$

or

(2) $$E = pc$$

Many derivations of the above equations are from Einstein's mass-energy equation, resulting in a circular dependency, which makes the derivations invalid. Instead, the recognition and theoretical substantiation of the existence of light momentum and pressure predates the publication of the mass-energy equation and is grounded in empirical observations. Material entities are composed of charged particles, and when electromagnetic waves, including visible light, incident on such an object, they exert forces on the charged particles per the Lorentz force. The energy and momentum of electromagnetic waves are then transmitted through these forces, and these forces perform work on the particles, augmenting their energy. This forms the foundation of the relationship between momentum and energy in electromagnetic waves. Here is a derivation of Equation (1) using a simplified form of the Lorentz force. On the other hand, the momentum (*p*) is defined as

(3) $$p = mv$$

Here, *m* is the mass of the entity and *v* is its velocity. In the case of a photon, traveling at the speed of light (*c*), the definition can be rewritten as

(4) $$p = mc$$

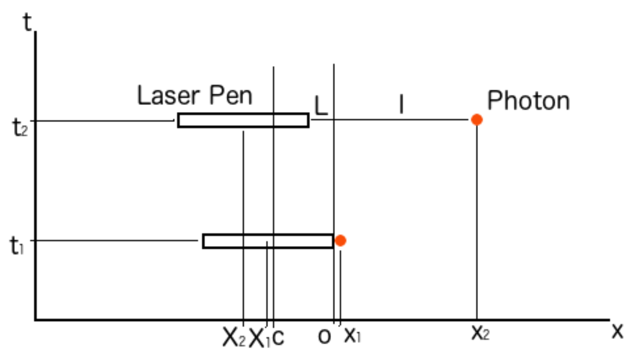
Replacing this momentum (*p*) in Equation (2), we found the mass-energy equation:

(5) $$E = mc^2$$

It is important to note that the transition from Equation (3) to Equation (4) represents a significant shift from classical to relativistic physics. The validity of Equation (4) will be further demonstrated through a thought experiment next.

Proof of p = mc for Photons

This proof is based on a thought experiment similar to Einstein's. Consider a laser pointer at rest in an inertial frame, as illustrated in the figure below, with the x-axis representing horizontal distance and the vertical-axis representing time. At time t_1, the laser emits a photon to the right along the x-axis. Since light has been experimentally shown to exert pressure and carry momentum and an effective mass, the emission causes the pointer and photon to recoil in opposite directions. By time t_2, the pointer has moved *L* meters to the left, and the photon has traveled *l* meters to the right, both measured relative to their original position at point *o*.



The mass center of both the pointer and the photon was initially at location *C,* which can be calculated like this.

(6) $$C = \frac{MX_1 + mx_1}{M+m}$$

Here, *M* and *m* represent the effective masses of the pointer and photon, respectively, and X_1 and x_1 denote the mass centers for the pointer and photon. Similarly, the new mass center for both the pointer and photon at time t_2 can be calculated as

(7) $$C = \frac{MX_2 + mx_2}{M+m} = \frac{M(X_1 + L) + m(x_1 + l)}{M+m} = \frac{MX_1 - ML + mx_1 + ml}{M+m}$$

Here, X_2 and x_2 represent the center of mass positions of the laser pointer and the photon, respectively, at time t_2. Treating the laser pointer and photon as a single isolated system, and assuming no external forces act on it, the center of mass of the system must remain unchanged. Consequently, the center of mass positions calculated in Equations (6) and (7) must coincide. This leads to the following equation:

(8) $$\frac{MX_1+mx_1}{M+m} = \frac{MX_1-ML+mx_1+ml}{M+m}$$

This can be simplified to

(9) $$ML = ml$$

By dividing the time duration (t_2-t_1) from both sides of the equation, it becomes

(10) $$\frac{ML}{t_2-t_1} = \frac{ml}{t_2-t_1}$$

Recall that the speed of the laser pointer (V) equals $L/(t2-t1)$, and the speed of the photon (v) equals $l/(t2-t1)$. The equation can be further reduced to

(11) $$MV = mv$$

This equation represents the conservation principle of momentum. Keep in mind that l is the distance the photon traveled during the time interval (t_2-t_1), so the speed (v) in Equation (11) is the speed of light (c). Therefore, the equation can be rewritten as

(12) $$MV = mc$$

The left side of the equation represents the pointer's momentum ($p_{pointer}$), which gains from the interaction with the photon. Thus, its value must equal the photon's momentum (p_{photon}). Therefore, we can establish the following relationships:

(13) $$p_{photon} = p_{pointer} = MV = mc$$

This proves Equation (4) ($p = mc$), confirming that definition (3) can be extended from Newtonian to relativistic physics. Therefore, the derivation in the previous section is valid.

Effective Mass

Photons do not have mass. Can we still use the momentum Equation (4) in the derivation? When people claim a photon has no mass, they are referring to the rest mass, which is typically denoted by m_0. The mass in this context refers to the effective mass. In the case of a photon, there is no rest mass, but it still possesses effective mass. The effective mass in the context of mass-energy equivalence is defined as

(14) $m = \frac{E}{c^2}$

To resolve the ultraviolet catastrophe problem, Planck postulated that the energy of electromagnetic waves must take the form of

(15) $E = hf$

Here, *h* denotes Planck's constant, and *f* is the wave frequency. This idea was later supported in Einstein's experiments for the photoelectron effect. Hence, based on definition (14), the effective mass of a photon should be

(16) $m = \frac{hf}{c^2}$

Some literature refers to the mass in the mass-energy equation (5) as a relativistic mass since the equation was initially derived from Einstein's theory of relativity. In that context, the relativistic mass (m_r) is given as

(17) $m_r = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$

Here, *v* denotes the velocity of a moving object. However, this definition imposes limitations on the applicability of the mass-energy equivalence principle. For an object with a nonzero rest mass ($m_0 \neq 0$), the effective mass (m) can be equated with the relativistic mass (m_r). The relativistic mass exceeds the rest mass for a moving object, reflecting an increase in mass due to its kinetic energy. In the case of a photon, the effective mass is given by Expression (16), which represents mass arising from its radiative energy. Since a photon has zero rest mass and moves at the speed of light *c*, directly applying the relativistic mass formula (17) yields the indeterminate form 0/0. This presents a challenge when attempting to extend the mass-energy equivalence to photons using that approach.

To serve as a universal principle in physics, the mass-energy equivalence must apply to all systems, regardless of whether they possess rest mass. The concept of *effective mass*, introduced in Equation (14), offers a well-defined framework that naturally extends to massless particles such as photons, thereby unifying the mass-energy relationship across all physical systems. The effective mass of a system accounts for contributions from all forms of energy—kinetic, potential, and radiative. As will be shown later, even rest mass is a relative quantity that depends on the observer's frame of reference.

Generalities of Physical Principles

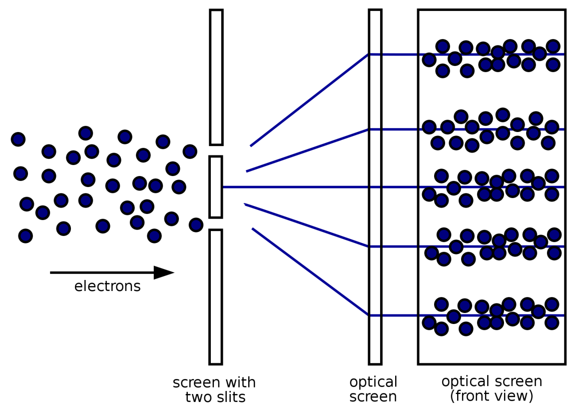
Electromagnetic waves, or photons, exhibit wave-particle duality. French physicist Louis de Broglie was wondering whether matter would also display a similar duality. The wavelength (λ) of a photon can be derived from Equations (1) and (15):

(18) $\lambda = \frac{h}{p}$

Inspired by this relation, de Broglie postulated that the wavelength of matter could be determined by substituting the momentum from Equation (3) into the above expression:

$$(19) \qquad \lambda = \frac{h}{mv}$$

To verify de Broglie's hypothesis, double-slit experiments were conducted with electrons, and the results revealed an interference pattern characteristic of wave behavior, as shown in the figure below, supporting de Broglie's postulation.



Building on de Broglie's concept of matter waves, Schrödinger formulated an equation that describes their behavior, paving the way for the development of quantum mechanics.

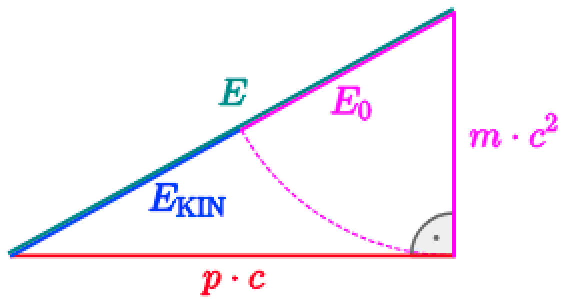
Since physical principles reflect certain general properties, we would expect that a photon's momentum can be expressed not only in the form of Equation (1) but also in the form of Equation (4). By combining these two equations, we arrive at the same definition for effective mass:

$$(20) \qquad m = \frac{E}{c^2}$$

This definition seems to emerge from the same set of equations, but it is based on a different concept: the general nature of momentum, which allows the application of the momentum definition in any system. This generality will be further explored in the next section, particularly in relation to the mass-energy equation and the energy-momentum relation.

Energy-Momentum Relation

The energy-momentum relation is the extension of mass-energy equivalence for many-body systems with nonzero momentum, which is formulated based on Einstein's Triangle



(21) $E^2 = (pc)^2 + (m_0c^2)^2$

This relationship aligns with the mass-energy equation, applicable to systems with or without rest mass. For a massless system ($m_0 = 0$), it takes the same form as Equation (2). For a system with a nonzero rest mass ($m_0 \neq 0$), the momentum term $(pc)^2$ relates to the system's kinetic energy. In the case of a stationary system ($p = 0$), it reduces to the mass-energy equation:

(22) $E = m_0c^2$

Indeed, for a single-body system, Equation (21) simplifies to Equation (5) by substituting the momentum term (p) from Equation (3) and the mass term (m) from Equation (17). This demonstrates that the energy-momentum relation is more general than the mass-energy equation, as it applies to systems involving multiple bodies.

There is, however, a significant difference between the kinetic energy referred to in the mass-energy equation and that in the energy-momentum relation. The kinetic energy in the mass-energy equation arises from an object's motion relative to the observer. In contrast, the kinetic energy in the energy-momentum relation can also be due to the sub-system's motion relative to the mass center of a system, such as a box of gas molecules. When an observer moves with the mass center of a system, there is no kinetic energy from the perspective of the mass-energy equation. In this context, the kinetic energy in the energy-momentum relation contributes to the internal energy and the rest mass of the system.

For example, consider the motion of Earth around the Sun. To an observer moving outside and alongside the solar system, this motion is considered internal to the solar system, and its kinetic energy contributes to the increase in the solar system's rest mass. However, to an observer at the Sun, the Earth's motion increases its own kinetic energy and effective mass. This illustrates that the concept of rest mass is relative, depending on the observer's frame of reference.

Energy-Mass Interchangeability

The fundamental principle illustrated by both the mass-energy equation and the energy-momentum relation is the interchangeable nature of mass and energy. While these equations might initially suggest that only kinetic energy exchanges with mass, this interchangeability extends to potential energy as well. For example, in hydrogen fusion, when hydrogen nuclei combine to form a helium nucleus, the mass of the system is reduced due to the release of a substantial amount of potential energy between hydrogen nucleons.

More directly, the energy contained in matter can be fully released, reducing its mass to zero. During positron-electron annihilation, the entire energy of the particles is converted into radiative energy. Any fundamental particle can annihilate with its corresponding antiparticle, releasing the total energy of the mass. Conversely, in pair production, high-energy photons interacting near a nucleus can create an electron and a positron, forming matter from pure energy.

Thus, effective mass encompasses not only kinetic, potential, and radiative energy but also reflects their ability to transform into one another. This interchangeability highlights a fundamental symmetry between mass and energy. Since energy is a conserved property of a system, effective mass must also be conserved. For further discussion on mass, see the article *Fundamental Problems about Mass*.

Revision History

- 06/08/2018: Initial Post on Stanford Site
- 11/02/2025: Published on Zenodo
- 12/17/2025: Adding Links to Summaries of Related Articles

Links to Summaries of Related Articles

- https://cs.stanford.edu/people/zjl/abstract.html, PDF
- https://sites.google.com/view/zjl/abstracts, PDF
- https://xenon.stanford.edu/~zjl/abstract.html, PDF
- https://doi.org/10.5281/zenodo.17967154, PDF

Further Literature

- Misconceptions in Thermodynamics (PDF: DOI) (中文: DOI)
- The Mechanism Driving Crookes Radiometers (PDF: DOI) (中文: DOI)
- The Cause of Brownian Motion (PDF: DOI) (中文: DOI)
- Can Temperature Represent Average Kinetic Energy? (PDF: DOI) (中文: DOI)
- The Nature of Absolute Zero Temperature (PDF: DOI) (中文: DOI)
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- Entropy Can Decrease (PDF: DOI) (中文: DOI)

- The Restoration Principle (PDF: DOI) (中文: DOI)
- Is There a Sea of Free Electrons in Metals? (PDF: DOI) (中文: DOI)
- Electron Tunnel (PDF: DOI) (中文: DOI)
- Unified Theory of Low and High-Temperature Superconductivity (PDF: DOI) (中文: DOI)
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- The Evolution from the Law of Gravitation to General Relativity (PDF: DOI) (中文: DOI)
- The Simplest Derivation of $E = mc^2$ (PDF: DOI) (中文: DOI)
- How to Understand Relativity (PDF: DOI) (中文: DOI)
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